Colour Processing in Tetrachromatic Spaces

Uses of tetrachromatic colour spaces

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Abstract: We exploit the geometry of the 4D hypercube in order to visualize tetrachromatic images.

1 INTRODUCTION

Tetrachromatic images $i : N \times M \rightarrow [0, 1]^4$ are images where each pixel has four spectral components, each component giving information regarding the energy contents of the pixel in a given spectral band. We assume that each component value of a pixel occurs in the interval $I = [0, 1]$ and the total gammut of the possible colours a pixel can take can be modeled with the hypercube $I^4$, a 4D colour being a point $[w,x,y,z]$ of the hypercube. Two points of the hypercube are the black (or "schwartz") vertex $s := [0000]$, and the white vertex $w := [1111]$; a subset of the hypercube is $A := \{[t,1,1,1] : t \in [0,1]\}$, the achromatic segment between $s$ and $w$. See (Restrepo, 2012a) and (Restrepo, 2012b). Tetrachromatic images can be visualized by feeding the RGB channels of a projector or screen with 3 of the bands W, X, Y Z of the image, in one or several of the of the 3! possible ways of doing this.

2 GEOMETRY AND 4D COLOUR

The (tridimensional) boundary $\partial I^4$ of the hypercube $I^4 \subset \mathbb{R}^4$ has a rich geometrical structure; it consists of $24 \times 8 = 192$ solid cubes with 16 vertices, 32 edges, and 24 square faces. A colour $[w,x,y,z]$ is on $T := \partial I^4$ if at least one of its coordinates is 0 or 1; each solid cube consists of the points having a given coordinate at value either 0 or 1; for example, the cube $\{[w,x,y,z] \in I^4 : w = 1\}$, which we denote as $\{w = 1\}$. Indeed, we write $\partial I^4 = \{w = 0\} \cup \{w = 1\} \cup \{x = 0\} \cup \{x = 1\} \cup \{y = 0\} \cup \{y = 1\} \cup \{z = 0\} \cup \{z = 1\}$. $\partial I^4$ is a piecewise linear (PL) tridimensional sphere that can be homeomorphed to a more standard, round $S^3$.

In the 2-skeleton of the complex structure of $T = \partial I^4$, you find 24 PL 1-spheres (one per face), 8 PL 2-spheres and 3 PL Heegaard tori. Geometrically, these manifolds can be used to define an orientation of the points in the hypercube that, with corresponding coordinate systems, is used to define several types of hue for 4D colours.

2.1 Tint

To give spherical coordinates $(d, \Theta)$ to any point $p \in I^4$, denote the central point of the hypercube as $g = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, let $d$ be a measure of the distance between $p$ and $g$ (e.g. the max of the absolute values of the components of $p - g$), and let $\Theta \in T$ be the point where the ray from $g$ through $p$ leaves the hypercube. Call $\Theta$ the tint, or generalized hue, and call $d$ the colourfulness, or generalized saturation of $p$. In this sense, $T$ is the set of tints. Note that the vertices $s$ ("black") and $w$ ("white") are fully colourful and are tints.

2.2 Chromatic Hue

A pair of vertices of the hypercube is said to be a pair of opposing vertices if the coordinates of one are the "negated" version of the coordinates of the other, for example, [0000] and [1111], or [0101] and [1010]. Eight PL 2-spheres, that are dodecahedra of square faces, result by considering the faces that do not meet a given pair of opposing vertices. Each of these 2-spheres serves as an equatorial 2-sphere for $\partial I^4$; for our purposes, the most relevant is the one having as opposing vertices $s$ and $w$. Call it the chromatic dodecahedron $D = \{w = 0, x = 1\} \cup \{w =
The rhombic dodecahedron is a Catalan solid, i.e. a polyhedron that is dual to an Archimedean solid; in this case, to the cuboctahedron, which has 12 vertices, 24 edges, 8 triangle faces and 6 square faces; two triangles and two squares meet at each vertex.

The abc coordinates of the intersection of the ray from the center of the rhombic dodecahedron through the projection of a chromatic point, and the boundary of the rhombic dodecahedron, gives an alternate hue $\eta$. The distance from the center of the rombic dodecahedron to the projection point is a measure of chromatic saturation $\sigma$; also, the projection $[\lambda, \lambda, \lambda]$ on A of $[w, x, y, z]$ is given by $\lambda := \frac{w+x+y+z}{2}$, gives a measure of luminance. Thus $\sigma = \sqrt{w^2 + x^2 + y^2 + z^2 - 4\lambda^2}$. In this way an alternate colour space3 to that with the $p\mu$ triangle results.

### 2.4 Tori

The tint of a colour $p$ different from g is given by $\Theta = g + \chi(p - g)$ where $\chi = \frac{1}{\min(|w'|, |x'|, |y'|, |z'|)}$ where $w' = w - 0.5, x' = x - 0.5, y' = y - 0.5$ and $z' = z - 0.5$. The indexes $i$ of the coordinates $\Theta$ of $\Theta = (\Theta_0, \Theta_1, \Theta_2, \Theta_3)$ of value 0 or 1 indicate the cube $\Theta$ is at; for example, if $\Theta_1 = 0$, then $\Theta \in \{x = 0\}$.

A coordinate system for the points in an $S^3$ results by considering the Heegaard splitting of genus 1. It uses two angles and a “signed radius” $r \in [-1, 1]$, rather than the better-known, spherical coordinates of three angles. A Heegaard torus splits the 3-sphere into two open solid tori and their common boundary. Out of the 24 square faces, 16 faces can be chosen that together are a Heegaard torus for $T = \partial 4^2$; this can be done in three ways since the 8 cubes in $T$

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1. This is computed by subtracting the average of the coordinates from each coordinate.

2. The rhombic dodecahedron is a Catalan solid, i.e. a polyhedron that is dual to an Archimedean solid; in this case, to the cuboctahedron, which has 12 vertices, 24 edges, 8 triangle faces and 6 square faces; two triangles and two squares meet at each vertex.

3. To get a B&W image from a color image, in the trichromatic case, it gives better visual results to use the max (as in the HSV colour system) than to use the average.

### Table 1: 14 vertices of the chromatic dodecahedron are projected onto the 3-subspace normal to $\{1,1,1,1\}$. Then, the projections are given 3-space coordinates in the third column.

<table>
<thead>
<tr>
<th>vertex</th>
<th>projection</th>
<th>[a, b, c]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0111</td>
<td>$[1, 1, 1, 1]$</td>
<td>[-0.8660, 0, 0]</td>
</tr>
<tr>
<td>0010</td>
<td>$[1, 1, 1, 0]$</td>
<td>[-0.2887, -0.4082, 0.7071]</td>
</tr>
<tr>
<td>0111</td>
<td>$[1, 0, 1, 1]$</td>
<td>[-0.5774, -0.8165, 0]</td>
</tr>
<tr>
<td>0001</td>
<td>$[1, 1, 0, 1]$</td>
<td>[-0.2887, -0.4082, -0.7071]</td>
</tr>
<tr>
<td>0101</td>
<td>$[1, 0, 1, 1]$</td>
<td>[-0.5774, 0.4082, -0.7071]</td>
</tr>
<tr>
<td>0100</td>
<td>$[1, 0, 1, 1]$</td>
<td>[0.5774, 0.8165, 0]</td>
</tr>
<tr>
<td>0110</td>
<td>$[1, 1, 0, 1]$</td>
<td>[0.2887, 0.4082, 0.7071]</td>
</tr>
<tr>
<td>1010</td>
<td>$[1, 1, 0, 1]$</td>
<td>[0.5774, -0.4082, -0.7071]</td>
</tr>
<tr>
<td>1011</td>
<td>$[1, 1, 0, 1]$</td>
<td>[-0.2887, 0.8165, 0]</td>
</tr>
<tr>
<td>1001</td>
<td>$[1, 0, 1, 1]$</td>
<td>[-0.5774, -0.4082, 0.7071]</td>
</tr>
<tr>
<td>1000</td>
<td>$[1, 0, 1, 1]$</td>
<td>[0.2887, 0.4082, -0.7071]</td>
</tr>
<tr>
<td>1100</td>
<td>$[1, 1, 0, 1]$</td>
<td>[-0.5774, 0.8165, 0]</td>
</tr>
<tr>
<td>1101</td>
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<td>[0.2887, 0.4082, -0.7071]</td>
</tr>
<tr>
<td>1110</td>
<td>$[1, 1, 0, 1]$</td>
<td>[0.5774, 0.8165, 0]</td>
</tr>
<tr>
<td>1100</td>
<td>$[1, 1, 0, 1]$</td>
<td>[-0.5774, 0.8165, 0]</td>
</tr>
<tr>
<td>1000</td>
<td>$[1, 0, 1, 1]$</td>
<td>[0.2887, 0.4082, 0.7071]</td>
</tr>
</tbody>
</table>

The table shows the projection of each vertex onto the 3-subspace normal to $\{1,1,1,1\}$. The columns correspond to the vertex, its projection onto the 3-subspace, and its 3-space coordinates.
can be grouped in $\frac{1}{2} \binom{4}{1} = 3$ ways, into two groups of four cubes each, so that each group is a solid torus. Here, we consider the solid tori $V_{2z} := \{z = 0\} \cup \{y = 1\} \cup \{z = 1\} \cup \{y = 0\}$ and $V_{rx} := \{w = 0\} \cup \{x = 1\} \cup \{w = 1\} \cup \{x = 0\}$.

The boundaries of $V_{rx}$ and $V_{rc}$ are the torus $H; H$ can be seen as the union of four square pipe segments in two ways; each pipe segment (topological cylinder or annulus) is a stack of 1-squares that are meridians for the solid torus in question and longitudes for the other solid torus. For the solid torus $V_{2z}$ we have the pipes of square meridians with vertices

$$p_0 := ((0,0,0,0),(0,0,1,0),(1,0,0,0),(1,0,1,0), s \in [0,1]) \{x=0\},$$

$$p_1 := ((0,0,1,0),(0,1,0,0),(0,0,1,0),(1,0,1,0), s \in [0,1]) \{y=1\},$$

$$p_2 := ((0,0,1,0),(0,1,0,0),(1,0,0,0),(1,0,0,1), s \in [0,1]) \{z=1\}$$

 similarily, the boundary of the $V_{rx}$ is given by the pipes of square meridians with vertices

$$q_0 := ((0,0,0,0),(0,1,1,0),(0,1,0,1),(1,0,0,0),(1,1,0,0), s \in [0,1]) \{x=0\},$$

$$q_1 := ((0,1,0,0),(1,0,0,0),(0,1,1,0),(1,1,1,0), s \in [0,1]) \{y=1\},$$

$$q_2 := ((1,1,0,0),(1,1,1,0),(1,1,1,1), s \in [0,1]) \{z=1\} \{w=0\},$$

$$q_3 := ((1,1,0,0),(1,1,1,0),(1,1,1,1), s \in [0,1]) \{z=1\} \{w=1\}.$$ (1)

As we remarked above, $H = \cup P_i = \cup Q_i$. Each point of $T$ is either in the open solid torus $T_{ex}$, in the open solid torus $T_{cc}$, or in their common boundary $H$. The subindex $n$ of the pipe segment together with the value of $t$ or $s$, as in n.t., or n.s., gives an angular measure that ranges from 0 to 4, mod-4.

For $\Theta$ in an open torus, there is a distance $r \neq 0$ from the boundary of the solid torus the tint is at; the distance from the boundary is measured with the product metric; that is, for example, for the piece of solid torus bounded by pipe $P_0$, a tint point $[w,x,t,0]$ is at distance $0.5 - \max\{w - 0.5, |x - 0.5|\}$ from its boundary. Also, there are two 1-squares in pipes say $P_m$ and $Q_m$ with corresponding parameters $s$ and $t$ such that one of them (a meridian) bounds a two-square the tint is in, and the other intersects the first 1-square at a point $u$ on $H$ that is closest to $\Theta$. Let $u = (\phi, \psi)$ be a point in $H$, then $r = 0$. Denote $\Theta$ as $\theta \in (\phi, \psi, r)$, with the understanding that if $r = \pm 0.5$ (i.e. if $\Theta$ is precisely on the axis or core of a solid torus), exactly one of the angles $\phi$ or $\psi$ is undefined and only the longitude of the corresponding solid torus that contains $\Theta$ is needed and a coordinate corresponding to the meridian is left undefined. For example, the tint of $[0.9, 0.2, 0.3, 0.4]$ is $[1, 1/8, 1/4, 3/8] = (3.625, 2.875, 0.25)$, corresponding to pipes $P_3$ and $Q_3$, with $s = 5/8$ and $t = 7/8$.

2.5 Spinning

We generalize Artin’s concept of spinning is spinning with an $S^1$ to spinning with a sphere $S^n$. Given a subset $E$ of $R^2$ (such as the $p\mu$ triangle) with a closed subset $F$ (such as the $\mu$ edge), form the topological space $(E \times S^n)/\approx$, where each set of the form $\{f\} \times S^n$, $f \in F$, is identified to a point. Artin’s method provides a geometric embedding of subsets $F$ of $R^3$, in $R^4$, as 

$$\{x, y, z \cos \theta, z \sin \theta \}: f = (x, y, z) \in F, \theta \in [0, 2\pi) \}.$$  

2.6 Runge Ball

A 4D round space is obtained by deforming the hypercube into the standard 4-ball $\{w', x', y', z' \in R^4 : w^2 + x^2 + y^2 + z^2 \leq 1\}$. This can be done in several ways; one is to spin the $p\mu$ triangle, deformed to a semicircle, around $S^2$, with hinge the $p$ axis of the triangle, where $S^2$ is derived from the chromatic dodecahedron; another is to spin the midpoint (that that originates at intermediate gray) with $S^1$, with hinge the point of intermediate gray. In the first case we have a space with coordinates the luminance, the chromatic saturation and a 2D (the equatorial sphere derived from the chromatic dodecahedron) spherical hue; in the second case, we have a space with coordinates given by the generalized saturation $r$ and a generalized 3D hue given by the $S^1$ that is derived from the boundary of the hypercube.

Let $[w, x, y, z]$ be a point in the hypercube, shift the hypercube so that intermediate gray ends up at the origin of 4-space $R^4$ and rescale so that the maximum values of the coordinates is 1 and the minimum is -1. Let $\lambda = \max\{w', |x'|, |y'|, |z'|\}; \lambda \neq 0$, the point on the boundary of the hypercube that is in the same direction is $\lambda^{-1}(w', x', y', z')$ (at least one of its coordinates has value of 1); let $d = \lambda^{-1}(w^2 + x^2 + y^2 + z^2)$ and normalize by this length (with the result that the hypercube is deformed into a 4-ball), getting the point $s = [x_0, x_1, x_2, x_3] := \lambda^{-1}(w', x', y', z')$ whose distance from the center of the ball is
\[
\kappa = \sqrt{w^2 + x^2 + y^2 + z^2} = \Lambda. \text{ Thus } \\
\kappa = \max \{2w - 1, 2x - 1, 2y - 1, 2z - 1\} \text{ is the colourfulness of the point } \left[w, x, y, z\right]. \chi = \frac{1}{\kappa}.
\]

3 PROCESSING

By colour processing a digital tetrachromatic image, we mean the application of a law to each pixel in the image, producing a new tetrachromatic image. The image is then to be visualized or fed to a computer vision algorithm. By appropriately modifying the hue, it is possible to visualize tetrachromatic images in such a way that certain aspects are made conspicuous.

The linear (i.e. noncircular, nonspherical) coordinates such as colourfulness, chromatic saturation and luminance, are transformed via exponential-law maps \(x^\gamma\). The hue may be independently processed by automorphisms either of the 3-sphere, a hue sphere or of a hue torus. As the hue surfaces are rotated or otherwise automorphed, the colours of a tetrachromatic image may change in interesting ways when trichromatically visualized. The automorphisms respect the continuity; the rotations are isometries and respect the antipodicity or complementary colours as well. The simplest modification type of the hue of 4D torus may change in interesting ways when automorphed either of the 3-sphere, a hue sphere or the Heegaard torus. As the hue surfaces are rotated or equivalently, the rotations of \(\mathbb{R}^4\) can be coded as a pair \(\left(\theta_1, \theta_2\right)\) in \(\mathbb{S}^3 \times \mathbb{S}^3\) in the sense that a unit quaternion is being pre and post multiplied by unit quaternions. The space \(\mathbb{H}\) of the quaternions can be seen as \(\mathbb{R}^4\) or as \(\mathbb{C}^2\). For \(\mathbb{C}^2\), the analogous case of an orthogonal transformation is that of a unitary transformation that, rather than preserving the structure of the inner product in \(\mathbb{R}^2\), it preserves the standard hermitian form \((z_1, z_2).\left(w_1, w_2\right) = z_1 \bar{w}_1 + z_2 \bar{w}_2\). The set of unitary transformations has the group structure \(\text{SU}(2)\). A point of \(\mathbb{S}^3\) can be denoted as a pair \((z_1, z_2) \subset \mathbb{C}^2\) with \(z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\).

For toroidal hue, for PL rotations, the 1D squares

\[p = [1/2, 1/2, -1/2, -1/2], q = [-1/2, -1/2, 1/2, 1/2], \gamma = 0.6; \text{ bands } 1 \text{ (in } \mathbb{R} \text{), } 3 \text{ (in } \mathbb{G} \text{), } 4 \text{ (in } \mathbb{B} \text{).}, p = [1/2, 1/2, 1/2, -1/2], q = [-1/2, -1/2, 1/2, 1/2], \gamma = 1.0; \text{ bands } 1, 2 \text{ and } 3.

\[p = [1/2, 1/2, -1/2, -1/2], q = [-1/2, 1/2, 1/2, 1/2], \gamma = 1.0; \text{ bands } 2, 3 \text{ and } 4.

\[p = [1/2, 1/2, 1/2, -1/2], q = [1/2, -1/2, 1/2, 1/2], \gamma = 1.0; \text{ bands } 1, 2 \text{ and } 4.

with sides parallel to the axes \(w\) and \(x\) are meridians of the \(yz\) solid torus and lengths of the \(wx\) solid torus; the \(1D\) squares with sides parallel to the axes \(y\) and \(z\) are meridians of the \(wx\) solid torus and lengths of the \(yz\) solid torus. Similarly for the other cases. Shifts around such squares implement modifications of hue.

4 CONCLUSIONS

Tetrachromatic colour spaces find applications in the visualization of 4-spectral images. Its use in satellite imagery (Landsat, 2012) is very likely providing alternative ways to the mere feeding of the visualizing RGB channels with permutations of the image \(wxyz\) channels. Also, as a technique for computational photography, the exploitation of IR and UV bands is likely to be of use in different ways. Further work remains to be done in the exploration of automorphisms of spheres and tori different from isometries. Depending on the application different types of tetrachromatic colour processing will be needed.

REFERENCES
