Colour Processing in Runge Space

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ABSTRACT
We do colour image processing in an RGB-derived spherical space with colour attributes given by hue, colourfulness (as opposed to grayness and somewhat different from saturation) and lightness; we call it Runge space. This spherical space is as intuitive as more common spaces of the type hue-saturation-luminance (called here $\eta, \Sigma, \Lambda$ spaces), yet it avoids the continuity problems of the transformation $(R, G, B) \rightarrow (\eta, \Sigma, \Lambda)$ that result from normalizing the saturation by the luminance, or of having a geometrically nonhomogeneous space, when the saturation is left un-normalized. We give Matlab routines for the conversions between colour spaces RGB and Runge, and present applications of colour modification in Runge space.

Keywords: Colour processing, hue, saturation, luminance, double cone, Runge.

1. INTRODUCTION
Besides being close to camera and display colour coding, RGB space is a cube; this allows for independent changes of the components R, G and B which is an advantage when changing the colour of a pixel. This homogeneity property is not shared by many color spaces of the type hue-saturation-luminance (to which we refer to generically as $\eta\Sigma\Lambda$-type spaces).

Colour modification is made more intuitive by the use of $\eta\Sigma\Lambda$ spaces; however, the choice one such space is a delicate one as this type of space commonly normalizes the saturation component by the luminance component, making the transformation $RGB \rightarrow \eta\Sigma\Lambda$ a discontinuous one; this results in saturation artifacts for values of luminance near 0, near 1, or both which in turn makes it hard to change the color of a pixel along a given dimension (e.g. hue) regardless of the values of the colour along the other dimensions (e.g. luminance and/or saturation).

Useful RGB-derived measures of luminance $\Lambda$ and saturation $\Sigma$ are those given respectively by the midrange $\mu$ and the range $\rho$ of the triple $(R, G, B)$; it is known $^1$ that (assuming that the $RGB$ variates are $[0, 1]$-valued) the corresponding space is an isosceles triangle of height 1 (along the $\rho$-axis) and basis length 1 (along the $\mu$-axis). A double cone results by spinning the triangle around its base, with spin angle corresponding to the hue of the colour point. We call this space double-cone $\eta \rho \mu$ space. The main purpose for considering this space is the derivation of an important colour spherical space.

In two RGB-derived, spherical ”Runge” spaces (round solid balls) we describe below, the center of the ball corresponds to intermediate gray (with RGB coordinates $(1/2, 1/2, 1/2)$); the north pole to white and the south pole to black. The (nonplanar) hexagon of RGB cube having as consecutive vertices pure red = $(1 \ 0 \ 0)$, pure yellow = $(1 \ 1 \ 0)$, pure green = $(0 \ 1 \ 0)$, best cyan = $(0 \ 1 \ 1)$, pure blue = $(0 \ 0 \ 1)$ and best purple = $(1 \ 0 \ 1)$ is called the chromatic hexagon, see Fig. 1. Runge’s colourfulness is given by the distance to the center of the ball; the azimuth angle determines the hue while the inclination angle determines the lightness.

The possibility of independent changes of the attributes of a colour in a given space depends on the geometrical homogeneity of the corresponding coordinate transformation. In a sense, we are concerned with transformations $R^3 \rightarrow R^3$, with domain restricted to the positive unit cube. We say that a subset $A$ of $R^3$ is geometrically homogeneous with respect to the transformation $T : R^3 \rightarrow R^3$ if for intervals $I, J, K \subset R^1$ (not necessarily open or closed) $A = T(I \times J \times K)$. For example, in $R^2$, with respect to the transformation of polar coordinates, the

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geometrically homogeneous sets are disks, annulli and circular sectors and circular sectors of annulli; in $\mathbb{R}^3$, with respect to cylindrical coordinates, cylinders, cylindrical sectors and pie slices; wit respect to spherical coordinates, solid balls, spherical secyors, spherical cones, etc. The usefulness of such considerations lies in the fact that the coordinates $i$, $j$ and $k$, with $(i,j,k) \in I \times J \times K$, of $T(i,j,k) \in A$, can be changed independently and in any order without ever leaving the set $A$. We say that $I \times J \times K$ is the parameter set of $T(I \times J \times K)$.

The parameter set of Runge’s ball is a rectangular prism in $\mathbb{R}^3$ (with Cartesian axes $r$, $\theta$ and $\phi$) which makes it geometrically homogeneous; because of this, in Runge space, the attributes of a colour point, hue, colourfulness and lightness, can be manipulated independently.

The traditional dimension of the saturation of a colour, as an attribute derived from RGB space, consists of a distance measure to the achromatic segment of the RGB cube; similarly, the luminance of a colour is a measure of the distance to the black vertex of the cube. The saturation, when it is left unnormalized by the luminance, has the drawback that very light and very dark colours have a limited range of possible saturations, conditioning the possible changes of saturation of a pixel on its luminance; on the other hand, when the saturation is normalized by the luminance, since achromatic colours have null saturation while colors arbitrarily close to black are nearly fully saturated, saturation discontinuities near black result. In Runge space, instead of saturation we have the related attribute of colourfulness, the distance to intermediate gray; there, black and white are as colourful as any other color on the boundary of the ball (which corresponds in a one-to-one fashion to the boundary of RGB cube.)

We make a point of explicitly indicating when an angular magnitude is left undefined, for example, the hue of an achromatic colour; we ”define as ‘undefined’ ” the magnitude in such cases. For example, the hue of an achromatic colour is ”left undefined as ‘∗’ ”; in this sense, the hue space becomes $S^1 \bigcup \{∗\}$, where $S^1$ stands for the hue circle. The fact that in Matlab the hsv coordinates of black are $(0 \ 0 \ 0)$, hue=0 being also the hue of red, is a source of confusion. On the other hand, the spherical coordinate for angle of inclination $\phi$ (which we use in Runge space) is not defined at the origin, where $r = 0$; nevertheless, since each value of $\phi$ determines either a point (at the poles of the sphere) or a horizontal disk of the ball, we assume correspondingly that the $\phi$-value of the origin is $\pi/2$. The Runge coordinates of intermediate gray are hue = $\eta = ∗$, colourfulness= $\kappa = 0$, lightness= $\lambda = 0.5$, which, as an $\eta\kappa\lambda$ Runge triple is $(∗, 0, 0.5)$; those of black are $(∗, 1, 0)$ and those of white are $(∗, 1, 1)$.

Likewise, we consider important to explicitly indicate when we are dealing with a cyclic (or circular) magnitude, e.g. the hue. In the cases where the hue is coded as an angle mod-$2\pi$, we write $\eta \in [0, 2\pi)^{°}$ (the little circle indicating the cyclic nature of the magnitude) and, if hues are numbers mod-1 (as in HSL space), we put $\eta \in [0, 1)^{°}$. Remember that averages, medians, etc. for cyclic or circular magnitudes are defined in ways different from their linear counterparts$^2$.\(^3\)

In addition to the introduction of the spaces $\rho\mu$ and Runge, we present a set of tools for the modification of the colour attributes in each space. In particular, we consider modifications of colour as they relate to changes un the intensity of the illumination. We modify the hue while taking care of the Bezold-Brücke effect, we modify the saturation taking into account the fact that at very large and at very low illumination levels usually the colourfulness is lost and we use power laws for the correction of both the lightness and the colourfulness.$^4$

2. MEANINGFUL SUBSETS OF THE RGB CUBE

Call the origin $(0, 0, 0)$ of the RGB cube pure black, the point $(1, 1, 1)$ pure white, the line segment between them the achromatic segment $\Phi$ and the plane through the origin and orthogonal to $\Phi$, the chromatic plane $\Pi$. Call the faces of the cube with points with $\min(R,G,B) = 0$, the dark corner of the cube, and those with $\max(R,G,B) = 1$, the light corner. The star (i.e. a set of edges) of those edges of the cube with zero median is called the dark "Y" of the cube while the star of the edges with unitary median is the light "Y" of the cube. Call the polygon formed by the edges of the cube of points $(R,G,B)$ with $\min(R,G,B) = 0$ and $\max(R,G,B) = 1$, the chromatic hexagon, shown in green in Fig. 1. Finally, call each triangle that has $\Phi$ as one its sides and a point on the chromatic hexagon as the corresponding opposite vertex, a (constant-) hue triangle or a chromatic triangle, shown in red in Fig. 1. Colour points on $\Phi$ (called achromatic colours) have an (undefined) hue of $∗$; also, the chromatic colours on each chromatic triangle have the same hue.
2.1 The Geometry of the Orderings of the Triple (R, G, B)

There are six (=3!) possible orderings (or permutations) of the triple (R, G, B); that is, considering nonrepetitive assignments of values R, G, B to the functions min(R, G, B), median(R, G, B), and max(R, G, B), the triple (\text{max,med,min}) has six possible values. Label the permutations as \( P_0 = (R, G, B) \), \( P_1 = (G, R, B) \), \( P_2 = (G, B, R) \), \( P_3 = (B, G, R) \), \( P_4 = (B, R, G) \), \( P_5 = (R, B, G) \), and consider the cyclic ordering, or sequencing, of the permutations given by \{ ...P_0, P_1, P_2, P_3, P_4, P_5, P_0 ... \}; this ordering is obtained by transposing (i.e. swapping) the first pair in the triple (R, G, B), then the second pair, then the first, etc.; for the permutation \( P_0 \), med = G and the colours with the corresponding RGB triples are the oranges; for \( P_1 \), med = R and we have the cetrines, for \( P_2 \), med = B and we have the greenish cyans, for \( P_3 \), med = G and we have the bluish cyans, for \( P_4 \), med = R and we have the bluish purples, finally, for \( P_5 \), med = B and we have the reddish purples. See Figure 2.

Using barycentric coordinates (with reference to the tetrahedron with vertices pure black, pure red, pure green and pure blue) for the points (R, G, B) on the chromatic triangle having vertex (r, g, b) on the chromatic hexagon, one has \( (R, G, B) = \lambda_1(1, 1, 1) + \lambda_2(r, g, b) + \lambda_3(0, 0, 0) = \lambda_1(1, 1, 1) + \lambda_2(r, g, b) \). Thus, the points (R, G, B) on each chromatic triangle have the same ordering (\text{max,med,min}) and the set of the chromatic triangles with a given ordering \( P_i \) form a tetrahedron (of which the chromatic triangles are slices given by the chromatic triangles). In fact, each chromatic triangle determines a flag with pole given by the achromatic line and each such flag intersects the chromatic hexagon at exactly one point; in turn, each point of the cube, not on the achromatic line, belongs to exactly one such flag. Given a point (R, G, B) in 3-space (not necessarily on the cube, for example a point on the chromatic plane \( \Pi \)) not on the achromatic line, the point of the chromatic hexagon that is the intersection of the chromatic hexagon and the flag the point (R, G, B) belongs to is given by \( H = \left[ \frac{R-m}{M}, \frac{G-m}{M}, \frac{B-m}{M} \right] \) where \( m = \min\{R, G, B\} \) and \( M = \max\{R-m, G-m, B-m\} \).

2.2 A few words on chromatic diagrams

A chromatic diagram is a flat circle (or polygon) of hues where opposite points correspond to complementary colors (i.e. colors whose additive combination is achromatic). The hue circle as depicted in Fig. 2 is a chromatic diagram; in RGB color space, the primary (pure) red is not unique red and its addition to the primary G produces yellow (it is unique red and unique green that are complementary colors.) In this diagram, the hues of pure red, pure green and pure blue differ by \( 2\pi/3 \) rads. Usually, in a chromatic circle, consecutive hues among the four uniques (red, yellow, green, blue) differ by \( \pi/2 \). It is customary to work with a nonunique (e.g. spectral) red primary; however, for the purposes of color modification one may consider otherwise. The use of a unique red primary requires a nonuniform circular shift \( S^1 \rightarrow S^1 \) of the variable \( \eta \).
Figure 2. The cube is partitioned into six tetrahedra according to the orderings of the triple \((R, G, B)\). A corresponding labeling \(\{0, 1, 2, 3, 4, 5\}\), of the regions of \(\beta\)-space RGB, on the chromatic plane \(\Pi\) is indicated.

Figure 3. Schematic representation of the min, median, max, range and cuasirange.

### 3. HUE-SATURATION-LUMINANCE TYPE SPACES

#### 3.1 \(\eta\sigma\lambda\)-type spaces

The \(\eta\) (hue), the \(\Sigma\) (saturation or colourfulness) and the \(\Lambda\) (luminance) variables can be derived from RGB values in several ways, as in the spaces HSV, HSL and HSI. Usually, \(\eta\sigma\lambda\)-type spaces normalize the saturation component by the luminance component where the \(\Sigma\Lambda\) space is a subset of a rectangle and the \(\eta\Sigma\Lambda\) space is correspondingly a subset of a cylinder; this, however, results in saturation artifacts at luminance values near 0, near 1, or both. Also, it may so happen (e.g. in HSI space) that the space is not a complete cylinder and a careless modification of the color of a pixel may result in an invalid point on the cylinder.

Geometrically, the luminance of a colour point in RGB cube is a measure of its distance to the origin while the saturation is a measure of the distance to the achromatic line and the hue is a measure of the angle that is measured on the projection on the plane (which is parallel to the chromatic plane \(\Pi\)) that contains pure red \(R\), pure green \(G\) and pure blue \(B\), with axis the achromatic line, and measured with respect to the projection of \(R\). In \(\eta\sigma\lambda\)-spaces, the hue \(\eta\) is the variable most consistently defined. It may be visualized on the chromatic plane \(\Pi\) as the angle the rays that result as the projections of the chromatic triangles, make with the ray corresponding to pure red, positive in the direction of the ray of pure yellow.

#### 3.2 Caveats

Color spaces are tridimensional; they are usually cubes, cylinders and balls, or subsets of them. Coordinates are usually functions with domain (called here the parameter space) a rectangular prism in \(R^3\), or a subset of it. For example, with spherical \((r, \theta, \varphi)\) and cylindrical \((\rho, \theta, z)\) coordinates, a finite solid ball and a finite solid cylinder are images of rectangular prisms (not open nor closed) in the original 3 space. With polar coordinates \((\rho, \theta)\), the origin of the plane has no defined angle \(\theta\). Likewise, for a solid cylinder of radius 1 and height 2, centered at the origin, the parameter space is given by \((\rho, \theta, z) \in (0, 1] \times [0, 2\pi) \times [-1, 1]\), also a parallelepiped, to the corresponding, image space which is a cylinder minus its axis \((\rho = 0)\), we add the central vertical axis,
which includes the origin; the vertical axis has coordinate $\theta$ "left undefined" as $\ast$. For a solid ball of unit radius, centered at the origin, the parameter space is $(\rho, \theta, \phi) \in (0, 1] \times [0, 2\pi) \times (0, \pi)$, which is also a (not open, nor closed) rectangular prism. To the corresponding image space, we add the origin and the north and south rays. At the origin, $r = 0$ and the coordinates $\theta$ and $\phi$ are "left undefined", as $\ast$ and $\ast$, respectively; for the two open rays through the north pole (where $\phi = 0$) and through the south pole (where $\phi = \pi$) the hue coordinate $\theta$ is "undefined as" $\ast$. It turns out convenient to assume $\phi = 1/2$ at the origin, though.

HSL, HSI and HSV spaces are best described using cylindrical coordinates; nevertheless, even though the image spaces of the spaces have a shape that is approximately cylindrical, it is not really cylindrical and the corresponding parameter spaces are not complete square prisms. The typical problem is that the saturation-luminance $\Sigma \Lambda$ image space for a given hue $\eta_0$ is not a rectangle, rather, such slices may have other shapes and also shapes that depend on $\eta$. Also, the coordinate transformation may be discontinuous near black or near white or both.

This presents a drawback when one such space is being used for color modification. This results in the need of using a pre-specified ordering for an automated change of the values of $\eta$, $\Sigma$ and $\Lambda$ or in the need of performing further changes to obtain legal coordinates, within the actual image space. We avoid this without normalizing the saturation component, with the use of the spherical spaces described below in Section 6. For this, we give a special role to intermediate gray; also, we use of gruniness (nearness to intermediate gray) as opposite of colourfulness instead of the use of saturation; in fact, we say that pure black and pure white are both grayness this, we give a special role to intermediate gray; also, we use of normalizing the saturation component, with the use of the spherical spaces described below in Section 6. For performing further changes to obtain legal coordinates, within the actual image space. We avoid this without need of using a pre-specified ordering for an automated change of the values of $\eta$.

### 3.3 The spaces HSL, HSI, HSV and $\eta \mu \nu$

In the spaces HSV, HSI and HSL, the points on the dark corner, different from pure black, have saturation $\sigma = 1$ and, since the saturation of pure black (and each point on the achromatic line) is defined to be 0, one has a discontinuity of the corresponding transformations from RGB space.

The range $\rho(R,G,B) = \max(R,G,B) - \min(R,G,B)$ and the midrange $\mu(R,G,B) = \frac{\max(R,G,B) + \min(R,G,B)}{2}$ of the triple $(R,G,B)$ are respectively measures of saturation and of luminance that we use below in Section 5. We denote as $\rho_1$ the quasirange $\max(R,G,B) - \min(R,G,B)$ and we denote as $\mu_2$ the "upper midrange" $\frac{\max(R,G,B) + \text{median}(R,G,B)}{2}$.

#### 3.3.1 The space HSL

The luminance component $\lambda$ of HSL colour space is given by the midrange: $L = \mu$.

The saturation component $\sigma$ is defined as $S = \frac{\sigma}{\rho}$, if $0 < \mu \leq 0.5$ and $S = \frac{\pi(1 - \mu)}{2}$, if $1 > \mu > 0.5$.

The circular hue component $\eta$ is coded in the interval $[-\frac{1}{6}, \frac{5}{6})$ (the little circle indicates that the numbers are to be taken mod-1), depending on the ordering of the triple $(R,G,B)$; it is defined as $H = \frac{R-G-B}{\rho}$ for the orderings $P_5$ and $P_0$: $H = \frac{1}{6} \frac{B-R}{\rho} + \frac{1}{3}$, for the orderings $P_1$ and $P_2$, and $H = \frac{1}{6} \frac{R-G}{\rho} + \frac{2}{3}$, for the orderings $P_3$ and $P_4$.

For constant values of the luminance component $\mu$, near 0 and near 1, a very small change of $\rho$ results in a large change of HSL’s saturation $S$.

#### 3.3.2 The space HSI

For the space HSI, the luminance component is given by the projection $[I, I, I]$ of the colour point $C = [R,G,B]$ on the line $\Phi$, where $I = \frac{R + G + B}{3}$. The projection of the colour point on the plane $\Pi$ is given by $P_{\Pi}(C) = \frac{1}{3} [2R - G - B, 2G - R - B, 2B - R - G]$.

The hue component $\eta$ is defined as the angle that the projection $P_{\Pi}(C)$ makes with the projection $[2/3, -1/3, -1/3]$ of pure red. The cosine of the hue is then given by $\alpha := \cos(\eta) = \frac{2R - G - B}{2\sqrt{R^2 + G^2 + B^2 - RG - RB - GB}}$, which can be derived as the cosine in a dot product. Alternatively, writing $P_{\Pi}(C) = [a, b, -(a + b)]$, we have $C = [I, I, I] + [a, b, -(a + b)]$ and the alternate expression $\alpha = \frac{\sqrt{3}a}{2\sqrt{a^2 + b^2 + ab}}$ results. One gets $\eta = \arccos(\alpha)$, if
$G \geq B$ (orderings $P_0$, $P_1$ and $P_2$ of the triple $(R,G,B)$), and $H = -\arccos(\alpha)$ if $G \leq B$ (orderings $P_3$, $P_4$ and $P_5$).

The saturation component of the HSI space is given by $S := 1 - \frac{\text{min}(R,G,B)}{I} = \frac{2 \sqrt[3]{2} \text{min}(R,G,B)}{I}$. The set of values that the saturation-luminance pair $(S, I)$ can take, depends on the value of the hue component $H$. This posses problems since, for the modification of the hue at large values of $I$, it may be necessary to modify the saturation as well. For a given hue $H$, the possible values of the pair $(S, I)$ are bounded below by the line segment $I = 0$, with $S \in [0, 1]$; on the left, by the axis $S = 0$ with $I \in [0, 1]$, on the left and above, by the vertical segment \{(S, I) : S = 1, I \in [0, I_0]\} and a segment of a hyperbola \{(S, I) : I = \frac{I_0^3}{1 + (1 - I_0^3)S}, S \in [0, 1]\}; $I_0$ is indirectly a function of the hue $H$; more directly, it is a function of the median med’ of the point on the chromatic hexagon that is vertex of the chromatic triangle that contains the point, and we have $I_0 = \frac{1 + \text{med}}{3}$ see Figure 4.

3.3.3 The space HSV

For the space HSV, the saturation is given by $1 - \frac{\text{min}(R,G,B)}{\text{max}}$ and the luminance component is given by the $\text{max}$. The hue is the same as that for the HSL system. (In the three cases HSL, HSI and HSV, the hue is constant for the points on each chromatic triangle.) Even tough a geometrically uniform space (HSV range space is a cylinder) and the possible values of the pair $(\text{max}, \frac{\rho}{\text{max}})$ are those in $[0, 1]^2$, in this square, the ”lines of constant $\rho”$ are segments of a hyperbola connecting the points $(\rho, 1)$ and $(1, \rho)$. Each of these hyperbolas intersect the line $s = \text{max}$ at the value $\text{max} = \sqrt{\rho}$; thus, for values of $\rho$ close to 0, the distance from the origin of the square to the point of intersection grows rather abruptly with $\rho$. For small $\rho$, we are near the achromatic line, and we have a sharp decrease of saturation from 1 towards 0, as the max moves away from zero.

4. LOCAL VERSUS POINTWISE MODIFICATION

One has pointwise modification when the new value of a pixel variate depends only on the original value of the variable. One has local modification when the new pixel colour depends on the colours of the pixels in a window around the pixel. Thus, contrast enhancing is a local processing. In an extreme case, the window can be a large segment of the image. In the most extreme case when the window is the whole image, we turn back to pixelwise processing where there is a unique change rule for the whole image. Because of this, pointwise modification is also global modification.

5. DOUBLE CONE $\eta\rho\mu$ SPACE

A very basic $\eta\Sigma\Lambda$ colour space that avoids the discontinuity issues that arise from the normalization of the saturation by the luminance, is the space with hue component given by the angle $\eta$, with saturation component given by the range $\rho$ of the triple $(R,G,B)$ and with luminance component the given by the midrange $\mu$ of the triple.
The $\rho$-$\mu$ triangle is the image space of the transformation $(R,G,B) \mapsto (\mu, \rho)$; see Figure 5. Each chromatic triangle of the RGB cube maps in a bijective fashion to the $\rho$-$\mu$-triangle; in fact, each point on the chromatic hexagon is mapped to the vertex point $(\mu, \rho) = (0.5,1)$; the achromatic diagonal line of the cube is bijectively mapped to the base of the triangle. Points on the faces of the dark (resp. light) corner get mapped to the left (resp. right) edge of the triangle. By spinning this triangle around its base, with spin angle given by $\eta$, the double cone $\eta$-$\rho$-$\mu$ colour space is obtained. The saturation of double-cone space $\eta\rho\mu$ is not normalized by the luminance; this is so at the cost of not being parameter-homogeneous space, that is, the $\eta$-$\rho$-$\mu$ corresponding Cartesian coordinate space is not a rectangular prism but the Cartesian product of a rectangle and a triangle. This of course means that that the allowed range of possible saturations a color may have depends on the value of its luminance: very light and very dark colors must have a small saturation.

![Figure 5. Luminance-Saturation $\rho\mu$ triangle $\Delta$ (same for each hue angle) of $\eta\rho\mu$ double cone colour space (not drawn to scale). By spinning the triangle across the base of the triangle, a double cone results; values of corresponding HSV's saturation $\sigma = \rho/\max$ are indicated.](image)

Matlab code for the transformation from RGB space to $\eta\rho\mu$, and back, is shown below.

```matlab
function [A] = RGB2ERM(v)
M = [0.5+0.5/sqrt(3), -0.5+0.5/sqrt(3), -1/sqrt(3); -0.5+0.5/sqrt(3), 0.5+0.5/sqrt(3), -1/sqrt(3); ...
1/sqrt(3),1/sqrt(3),1/sqrt(3)]; %rotation matrix
zred = (M*v')'; %red transformed
etared = angle(zred(1)+i*zred(2)); % reference hue
z = (M*v');
if v(1)==v(2) & v(2)==v(3) & v(3)==v(1)
et = 1000; %eta undefined as 1000 if color is achromatic
else
et = angle(z(1)+i*z(2) - etared);
end
mx = max(v);
mn = min(v);
ro = mx - mn;
my = (mx+mn)/2;
A = [eta, ro, my];
end
```

```matlab
function [A] = ERM2RGB(w)
MIN = w(3) + 0.5*w(2); % mu=w(3), rho=w(2)
MIN = w(3) - 0.5*w(2); % if eta arrives "undefined"
if w(1) == 1000
end
```

A=[MIN MIN MIN];
else

orthonormal basis for chromatic plane PI; RR comes from projection of red; YY is orthogonal to RR
RR=[sqrt(2)/sqrt(3) -1/sqrt(6) -1/sqrt(6)];
YY=[0 1/sqrt(2) -1/sqrt(2)]; %point on PI that is also on plane that contains chromatic triangle of angle eta
PP= RR*cos(w(1)) + YY*sin(w(1)); %intersection with chromatic hexagon
PPP= PP - [min(PP), min(PP), min(PP)];
PP= RR*cos(w(1)) + YY*sin(w(1)); %intersection with chromatic hexagon
PPP= PP - [min(PP), min(PP), min(PP)];

point on RGB cube lives on chromatic triangle with vertices S, H and W; barycentric coordinates are LS, LH and LW; LH=rho and LW=min
C= w(2)*H + [MIN MIN MIN];
A=C;

More than a space for colour modification, we consider \( \eta \rho \mu \) double cone space as an intermediate step to obtain a spherical space where the chromatic hexagon of RGB space is mapped to the equator of a spherical colour space. Nevertheless, it is instructive to consider possible ways of modifying the colors of pixels in this (not parameter homogeneous) space.

### 5.1 Luminance and saturation modification in double-cone space

In double cone \( \eta \rho \mu \) space, the hue component \( \eta \) may be altered regardless of the other two components, without the risk of arriving to an illegal color code. However, when modifying the saturation \( \rho \) or the luminance \( \mu \), we must ensure that the resulting pair \((\mu, \rho)\) lives in the \( \rho \mu \) triangle. In the \( \rho \mu \) triangle, the geometric loci corresponding to a given (HSV’s) saturation \( \sigma = \rho / \max \), consists of two straight segment lines: \( \rho(\mu) = 2\sigma \mu \), for \( \mu < 0.5 \) and \( \rho(\mu) = 2\sigma(1 - \mu) \), for \( \mu > 0.5 \), that go from the origin \((\mu, \rho) = (0, 0)\) to a maximum of \( \rho = \sigma \) at \( \mu = 0.5 \) and back to 0 at \((\mu, \rho) = (1, 0)\), as shown in Fig. 5.

To change the luminance \( \mu \) of a color in double cone space one may opt for preserving the value of \( \sigma \); thus, given the values of the saturation \( \rho_1 \) and of the luminance \( \mu_1 \) of the color to be modified, compute the value of \( \sigma \) as \( \sigma = \frac{\rho_1}{\mu_1} \) if \( \mu_1 < 0.5 \) and \( \sigma = \frac{\rho_1}{2(1 - \mu_1)} \) if \( \mu_1 > 0.5 \) (if the luminance \( \mu \) is 0 or 1, the saturation \( \rho \) should remain unchanged at 0), and move along the path of constant \( \sigma \); thus, to a change \( \mu_1 \rightarrow \mu_2 \), there corresponds a change \( \rho_1 \rightarrow \rho_2 \), with \( \rho_2 = 2\sigma \mu_2 \) if \( \mu_2 < 0.5 \) and \( \rho_2 = 2\sigma(1 - \mu_2) \) if \( \mu_2 > 0.5 \).

To change the saturation \( \rho \), keep in mind that, for each \( \mu \), its minimum is 0 and its maximum is at the path corresponding to \( \sigma = 1 \). Thus, for luminances \( \mu \) different from 0 and 1 (in such cases the saturation \( \rho \) should be left unchanged at zero), one may choose to change the saturation \( \rho \) using an exponential law: that is, to use a function \( \rho \rightarrow 2\mu(\frac{\rho}{\mu})^\gamma \), if \( \mu < 0.5 \), and \( \rho \rightarrow 2(1 - \mu)(\frac{\rho}{2(1 - \mu)})^\gamma \), if \( \mu > 0.5 \); notice that, for \( 0 < \gamma < 1 \), the saturation is increased, while for \( \gamma > 1 \) the saturation decreases.

### 6. SPHERICAL SPACES: RUNGE-I AND RUNGE-II

We introduce two spherical spaces in what follows. In RUNGE-I, a ball of radius 1/2, more swiftly derived from RGB coordinates, the chromatic hexagon is mapped to a curved polygon on the boundary of the sphere that zig zags about the equator. In the RUNGE-II, of radius 1, passing through \( \eta \rho \mu \) space as an intermediate step, the chromatic hexagon maps bijectively to the equator of the ball. RUNGE-I transformation from RGB is computationally faster and less expensive.

We consider the spherical coordinates given by the azymuth \( \eta = \theta \), the inclination \( \phi \) and is the distance from the center \( r \). Using this convention, in both spaces we have Runge hue-colourfulness-lightness \( \eta \kappa \lambda \) color space, \( \eta \in [0, 2\pi] \) is the hue; the colourfulness \( \kappa \in [0, 1] \) is given by \( \kappa = 2r \) (\( \kappa = r \) in the case of RUNGE-II) and the lightness \( \lambda \in [0, 1] \) is given by \( \lambda = \frac{r - \phi}{\pi} \).
6.1 Runge-I

(Deforming the RGB cube.) To obtain the components of a colour point in the basic variant of Runge space, shift the cube so that the central point (intermediate gray) ends up at the origin, then, with center the origin, radially contract the cube to a ball of radius 1/2, finally, rotate the ball so that the achromatic diameter points upwards. The image space is a tridimensional ball (with spherical boundary) in $R^3$, centered at the origin, where the achromatic axis is oriented vertically with black at the south pole and white at the north pole. The colorfulness however differs from more common types of colour saturation since it is a measure of the distance from intermediate gray rather than a measure of the distance from the achromatic line; in fact, the colorfulness $\kappa$ is 0 only for intermediate gray and it has the maximal value of 1 for points on the surface of the ball, including black and white. The lightness is, as in the more common spaces, a measure of the distance of the colour point to pure black. The loci of $\lambda = constant$ is a point (black and white poles) when the constant is 0 or 1, and a horizontal disk in the ball for intermediate values of the constant.

Matlab code for the routines that convert RGB coordinates to Runge-I’s spherical $r - \theta - \phi$ color coordinates and back are given below.

```
function[A] = RGB2RUNGE(v)
M= [.5+.5/sqrt(3), -.5+.5/sqrt(3), -1/sqrt(3); -.5+.5/sqrt(3), .5+.5/sqrt(3), -1/sqrt(3);1/sqrt(3),1/sqrt(3),1/sqrt(3)];
x=[v(1)-0.5, v(2)-0.5, v(3)-0.5];
w= [0 0 0];
xx=[abs(x(1)), abs(x(2)), abs(x(3))];
if max(xx) == 0
w(1) = 0;
z= (M*x')';
else
k=max(xx)/sqrt(x(1)^2+x(2)^2+x(3)^2);
y=k*x;
z= (M*y')';
w(1)= sqrt(z(1)^2+z(2)^2+z(3)^2); %r
end
if x(1)==x(2) && x(2)==x(3) && x(3)==x(1)
if x(1)==0
w(2) = 1000; %undefined
w(3) = pi/2;
else
w(2)=1000; %undefined
end
else
w(2)= angle(z(1)+ 1i*z(2) ); %teta
w(3)= angle(z(3) + 1i*sqrt(z(1)^2+z(2)^2));
end
A=w;

function[A] = RUNGE2RGB(w)
M= [.5+.5/sqrt(3), -.5+.5/sqrt(3), -1/sqrt(3);-.5+.5/sqrt(3), .5+.5/sqrt(3), -1/sqrt(3);1/sqrt(3),1/sqrt(3),1/sqrt(3)];
z(3)= w(1)*cos(w(3)); %r cos phi
z(1)= (w(1)*sin(w(3)))*cos(w(2));
z(2)= (w(1)*sin(w(3)))*sin(w(2));
y= (M*x')';
if y(1)==0 & y(2)==0 & y(3)==0 x=y;
else
yy=[abs(y(1)), abs(y(2)), abs(y(3))];
k= sqrt(y(1)^2 + y(2)^2 + y(3)^2)/max(yy);
```
x=k*y;
color=[x(1)+.5, x(2)+.5, x(3)+.5];
end
A=color;

Figure 6. Image set of the chromatic hexagon in Runge-I space (in Runge-II, it is the equator of the ball).

6.2 Runge II

(Deforming the double cone:) We obtain a space, very similar to the one described in Section 6.1 by a somewhat
different procedure. Place the double cone so that its axis is vertically oriented, with the pure white pole pointing
upwards. Then, from intermediate gray, radially project its boundary outwards to a circumscribing sphere of
radius one and correspondingly, project the solid double cone to the solid ball, leaving the central point, inter-
mediate gray, fixed. (Deforming the double $\rho\mu$ triangle:) Alternatively, visualize this as follows; place the $\rho\mu$
triangle with its base vertically oriented, then radially project its boundary to a semicircle of radius one, then
spin the semicircle to get a sphere. Matlab code for the routines that convert RGB coordinates to Runge-II’s
spherical $r-\theta-\phi$ color coordinates and back are given below.

function[A] = RGB2RUNGEBIS(v)
%rotation matrix
M=[.5+.5/sqrt(3), -.5+.5/sqrt(3), -1/sqrt(3); ... -.5+.5/sqrt(3), .5+.5/sqrt(3), -1/sqrt(3); ... 1/sqrt(3),1/sqrt(3),1/sqrt(3)];
%Computes first rho mu space %compute reference hue angle
zred= (M*[1 0 0]'); %red rotated
etared = angle(zred(1)+ i*zred(2) ); % reference hue
%compute hue angle
z= (M*v')'; %rotate color point
if v(1)==v(2) & v(2)==v(3) & v(3)==v(1)
et=1000; %for achromatic colors eta is "undefined at 1000"
end
if v(1)==0.5
phi=1000;
else
eta = angle(z(1)+ i*z(2) ) - etared; %eta
end
mx= max(v);
nm= min(v);
ro= mx - mn;
my= (mx+nm)/2;
%shift triangle so that origin coincides with intermediate gray; make base of triangle of length 2.
X= 2*my-1;
Y= ro;
%triangle to vertical semicircle
if X==0 & Y==0
XX=0; ZZ=0;
else
if X > 0
k= (X+Y)/sqrt(X^2 + Y^2); %(X/(X+Y), Y/(X+Y)) is a point on the line y=1-x
else
k= (Y-X)/sqrt(X^2 + Y^2); %(-X/(X+Y), Y/(X+Y)) is a point on the line y=1-x
end
end
end
ZZ = k * X; %norm of new point is X+Y, or Y-X, which is 1, if on the triangle
XX = k * Y;
end
w(1) = sqrt(XX^2 + ZZ^2); %r
w(2) = eta; %theta
if X == 0 & Y == 0
w(3) = 1000; %=phi=1000
else
w(3) = angle(X + i * Y); %phi
end
A = w;

function [A] = RUNGEBIS2RGB(w)
r = w(1);
eta = w(2);
phi = w(3);
%pass from RUNGEBIS to ERM %compute contraction factor x from semicircle to triangle if phi == 1000
rho = 0;
mu = 0.5;
else %compute contraction factor using law of sines
if phi < pi/2
k = (1/sqrt(2))/sin(3*pi/4 - phi);
else
k = (1/sqrt(2))/sin(phi - pi/4);
end
rho = r * k * sin(phi);
X = r * k * cos(phi);
mu = (X + 1)/2;
end
v = [eta, rho, mu];
A = ERM2RGB(v);

6.3 Lightness and Colourfulness modification in Runge space

In Runge hue-colourfulness-lightness η-κ-λ spherical space, the hue is the usual circular variable η ∈ [0, 2π]◦
the colourfulness κ ∈ [0, 1] and the lightness λ ∈ [0, 1] are [0, 1]-valued and thus, amenable to exponential/law
corrections.

When modifying a color in RGB space, care must be taken so that each of the variates R, G and B remains
within the interval [0, 1]. Any map f : B^3 → B^3 from the 3-ball to itself describes a colour transformation
that can be applied globally to each of the pixels of an image; a set of such maps can be applied locally as well,
according to some criterion. Usually, we modify the three components separately. Power laws can be exploited
for the modification of the lightness and the grayness, while the modification of the hue requires of circular tools;
for example, the colourfulness component κ could be changed to κ^{1.2}, the lightness component λ to λ^{1.1}
and the hue contrast of an image can be locally enhanced e.g. by using the formula h_0 ← h_0 + α(h_0 − ̅h)mod − 2π,
where ̅h is the hue of the central pixel of the window, ̅h is a circular location measure (e.g. the circular mean
or the circular median) of the hues in the window, α is a control parameter, e.g., α = 1.5. "gamma correction"
or exponential laws, are C∞ automorphisms of [0, 1] that preserve orientation. When independently modifying
the variables λ and κ, since they are [0, 1]-valued, a function [0, 1] → [0, 1] is required. Usually, it is desired
to have an orientation preserving (increasing) homeomorphism (continuous bijection). An interesting family of
such functions is given by the power laws x → x^γ, γ > 0. Spherical spaces allow independent modification of
spherical coordinates so long as one respects the radius range [0, 1] and the range [0, π] of the inclination angle
φ; this can be achieved e.g. using power laws r → r^γ and φ → π(2π)^γ.
7. ON THE PROCESSING OF THE (CYCLIC) HUE VARIABLE $\eta$

For natural images, pixelwise hue processing includes hue shifts and, more generally, orientation preserving homeomorphisms of the circle which include Bezold Brücke effects, described below. Usually, achromatic pixels are left achromatic. Local hue processing tools on the other hand include contrast enhancing, e.g. unsharp masking, circular mathematical morphology and other tools. The circular average of the hues of two colors approximates the hue of the corresponding additive mixture; the farther away the hues are in the circle, the more desaturated the mixture will be, like if a bit of white were added. The circular average of hues $\eta_1, \eta_2, \ldots, \eta_N$ is given by $\sum_{n=1}^{N} \eta_n$ (where $\sum$ stands for the angle of the complex number $z$), provided $\sum_{n=1}^{N} e^{j\eta_n}$ is not the complex number $0 + 0j$, in which case the average is left undefined (if the average is being used as a smoothing tool, one might consider to change to achromatic the colour of the pixel in such a case).

7.1 Bezold-Brücke type effects

Bezold-Brücke effect says that in high intensity, oranges and cetrines become yellower while purples and cyans become bluer, and that, for low intensities, oranges and purples become redder while cyans and cetrines become greener; in both cases, the four uniques red, yellow, green and blue, remain constant. We assume here that the effect apples to related colors. Here we should take reference with unique red, opposite to green (assuming it is unique green.) In RGB space, pure blue [0 0 1] and pure yellow [1 1 0] are in fact complementary. For the purposes of implementing a Bezold Brücke effect, we assume that pure blue and pure yellow are uniques; in addition, we assume that the four uniques are evenly distributed (separated by $\pi/2$ radians). Thus, the hues of the uniques red, yellow, green and blue, in degrees, are given by $-30^\circ$, $60^\circ$, $150^\circ$ and $240^\circ$. Thus, we implement a BB effect with the formula $\eta \rightarrow \eta + \alpha \sin(\eta - \pi/6)$ where $\alpha$ controls the sense and amount of the effect.

![Unique hues diagram](image-url)

Figure 7. Unique hues: unique red (U): $-30^\circ$, unique yellow (UY): $600^\circ$, unique green (UG): $150^\circ$, unique blue (UB): $240^\circ$.

If we want to give a scene the appearance of sunset or dusk, we emphasize the reds and greens; while the appearance of an image under strong light noon conditions is given by emphasizing the yellows and blues.

As simple illustrations of possible colour modifications in Runge spaces, consider the jpeg images in Figures 9 - 11.
8. SATURATION/COLOURFULNESS CHANGES

Many times it is convenient to decrease the colourfulness at intermediate values, this is obtained with an exponential law with an exponent slightly larger than one. Although not a common technique, the colourfulness of a pixel can be made dependant on the lightness of the pixel. In fact, there is a tendency of colours to become unsaturated with high luminances, due to the fact that they tend to become whiter. Goethe pointed out to relationships between lightness (white) and yellow and darkness (black) and blue. An increase of colourfulness wets the objects and mixes them. A decrease of colourfulness dries the objects and makes them more independent, isolates them. For high dynamic images, we suggest to segment the image into clear and dark regions, correct the lightness with exponential laws and slightly increase the saturation in light regions, also with an exponential law. A small amount of positive Bezold Brücke effect may be appropriate as well.

![Figure 8. Hue shifts in hue circle, due to drastic increase (left) and decrease (right) of the intensity.](image)

9. CONCLUSION

We have introduced several colour spaces and shown their usefulness in colour modification. Spherical spaces are convenient for colour modification as they are geometrically homogeneous.

Regarding possible applications of colour modification, several pathways are still open,\textsuperscript{7,8} It is interesting to explore the interplay between colourfulness and lightness. Night vision in a city turns out to be a combination of scotopic rod-vision and photopic vision at points of the image where the intensity is high enough; it may be interesting to make achromatic in the image regions of luminance below a certain threshold to emulate this effect.
Lee et al.\textsuperscript{9} have argued in favor of hue correction as a function of brightness, for high performance displays; however, as the visual system will do some correction, the effect should not be overdone.

REFERENCES


Figure 11. Colour modification in RUNGE-II space of Image Ginebra, with \( \lambda^{0.8} \), colourfulness correction \( \kappa^{1.3} \) and Bezold Brücke with \( \alpha = 0.1 \).
